Combining pencil/paper proofs and formal proofs, a challenge for Artificial Intelligence and mathematics education

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Abstract We compare the pencil/paper proofs and formal proof of two traditional proofs in high-school geometry. We highlight the fact that slightly different formulations or proofs can lead to difficulties in the formalization. We discuss the challenges and impact on both mathematical teaching and on the design of AI tools for mathematical education.[1]

Introduction

The need for developing research at the interface between mathematics and computer science in education is growing due to the evolution of curriculum, in particular in France but also in many countries.

> "According to Howson and Kahane [Churchhouse et al., 1986], the relationship between mathematics and computer science – especially the influence of computer science in mathematics and the role of mathematics in computer science – is an epistemological and didactic issue that transcends school systems and national contexts. The use of computer tools in the teaching of mathematics and informatics, raises questions about the nature of these tools. This can be connected to the particular role played by mathematics in computer science, the proximity of some aspects of both disciplines and the common nature of some of their questions."
>
> [Durand-Guerrier et al., 2019, p 116]

Among these aspects, proof and logical issues are certainly among the most prominent. The second author has worked for long on the impor-

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[1] This is a preprint of the chapter of *Mathematics Education in the Age of Artificial Intelligence: How Intelligence can serve mathematical human learning.* edited by Philippe R. Richard, M. Pilar Vélez, Steven Van Vaerenbergh, Part II AI-supported learning of mathematics, Springer, 2021.

tance and interest of logical analysis of proofs for mathematics education [Durand-Guerrier, 2008]. As stressed by [Durand-Guerrier and Arsac, 2009, p. 148] logical analysis of proof for mathematics education fulfils several functions. The first one, which was the main purpose for logicians since the late nineteenth, is to check the logical validity i.e. the correctness of the proof. A second function is to understand the proving strategy of the author of the proof. A third one is to contribute to the understanding and appropriation of proofs, as part of the study of the contents of the course in which they appear and as a means to better understand what are mathematical proofs and their possible specificities in a given mathematical field on the other hand. In this respect, it contributes to conceptualisation. Different tools for realising such logical analysis have been considered, such as natural deduction or dialogical analysis, which allows intermediate formalisation between mathematical proofs and formal proofs. Moving to the design of AI tools for mathematics education such as proof assistant raises new questions.

In this chapter, we will highlight the issues raised by these new questions with two examples. The first one is the proof of the theorem that "the sum of angles of a triangle is two right angles" with a contrastive analysis of the Pythagorean proof as found in Euclid Elements, and a formalization of this proof in the Coq proof assistant[2]. The second one is around the proof of the so-called Varignon's theorem. The second author of the paper uses this theorem as basis for developing proof competencies in mathematics secondary teachers training. We will first motivate the choice of this theorem as food for thoughts on the didactical interest of combining pencil/paper proofs and formal proofs and compare several proofs of this theorem in different mathematical frames: Euclidean geometry, analytic geometry, vector geometry, the area method and their formalization in Coq. We then present the main features of the teacher training session around this theorem that consist in first providing individually at least two proofs of the theorem and to analyse them in line with the questions raised in the introduction, and then explore the "inverse" problem consisting in determining necessary and sufficient conditions for getting a rhombus, a rectangle, a square. We then discuss the interest of introducing in the training session a proof assistant in order to enrich the milieu of the situation.

[2] A proof assistant is software which allow the user of the system to state mathematical definition and properties and to prove theorems interactively using a formal language. The proofs are checked mechanically.

1 The sum of angles of a triangle is two right

1.1 Some questions and issues raised by the proof by Pythagoras

This is taken from [Durand-Guerrier and Arsac, 2009, p. 149-150] that was presented at the 19th ICMI conference on proof and proving in mathematics education and published in the pre-proceedings of the conference. We recall here the proof attributed to Pythagoreans that the sum of the angles in a triangle is equal to two right angles.

> "Given a triangle ABC, let draw DE parallel to BC through A. The alternate angles are equal, on the one hand the one under DAB to the one under ABC; on the other hand, the one under EAC to the one under ACB. Let add the one under BAC to the two others. The angles DAB, BAC, CAE, that means the ones under DAB, BAE, that means two rights, are hence equal to the three angles of the triangle. Hence, the three angles of the triangle are equal to two rights."

Two remarks stand out: 1. A first object is given, a triangle, and nothing is said about hypotheses; 2. A second object is introduced, a parallel line DE to BC through A, that appears as a key for the proof, due to the fact that the whole proof is built on properties of alternate angles. Then, two first questions emerge: Q1. What relationship between data and hypotheses? Do we use a hypothesis in this proof? Q2: What role for the introduction of objects? Could the main ideas of a proof be resumed to the list of objects that have to be introduced? In a middle school's textbook, we can read that it is necessary to take a triangle "absolutely ordinary (scalene)", that means that the proof deals with the general case. This leads to a new question: Q3. How is generality taken into account in Geometry? Is it the same process in Algebra? The proposition is on triangle, so it is natural to introduce a triangle. But the proof relies entirely on the introduction of a second object, a line. Introducing that line can be justified only by the proposition that "one line can be drawn parallel to a given line through any point not on the line". As a triangle is defined by a set of three points not on the same line, we can actually apply this statement. This allows us to answer to Q1: here the data are three points and the hypothesis is that they are not on the same line. So data are objects and hypotheses express relations between these objects. This was hidden in the initial writing of the proof where the necessity of using a hypothesis is masked by the material possibility of doing the construction: drawing a triangle, one determines three points not all on a same line, and then it is actually possible to draw the parallel. The proof could also be completed without relying on the hypothesis that the three points are not on the same line, but by performing a case distinction. This leads to a new question, closely related with Q3: what evidences are used in proofs, particularly in Geometry? And how are we sure to check validity? In fact, there are still other evidences hidden in that proof [Arsac, 1998]. The Pythagorean proof above provides an example of such evidence; we are inclined to conjecture that this recourse to evidence is possible because Geometry as a theory has been elaborated in such a manner that those types of evidence, that are expressed by true statements in the drawing register, are logically deducible in the theory (the axiomatic has been built on this purpose). However, it is also clear that some evidences in the drawing register have to be questioned in the theory, this corresponding to the back and forth between objects in an interpretation (here the drawing register) and the theory (here Plane Geometry), and hence between truth and validity [Durand-Guerrier, 2008]. Manders has argued that the use of diagrammatic inferences in Euclid is not a lack of rigor as it is restricted to specific statements about the relative position of geometric objects [Manders, 2011]. Several authors have proposed formal systems to provide a validity criterion for such inferences or diagrammatic inferences [Winterstein, 2004, Winterstein et al., 2000, Miller, 2007, Miller, 2012, Mumma, 2010, Avigad et al., 2009]. But, up to our knowledge, Euclid's proofs have never been checked using these formal systems. In the experience of the first author about the mechanical checking of Euclid's proofs of the first book of Euclid's elements [Beeson et al., 2019] and as will be demonstrated by the following formalization of the proof of Pythagoras, it is difficult to justify that diagrammatic inferences are not gaps in the proof because:

1. the diagrammatic inferences are hard to separate from other inferences because statements guarantying the relative position of geometric objects (what Manders calls co-exact attributes) often use as premises exact attributes
2. the diagrammatic inferences sometimes rely on properties which are not even visually evident, they are evident on an instance of the figure, but sometimes the genericity of the validity of the property relies on an exhaustive enumeration of the different possible figures.

On a pragmatic level, it is not possible to prove every "evidence" of the drawing register; hence, to know which "evidences" are (logically) acceptable in a proof is clearly a difficult question that necessitates both mathematical knowledge and logical competencies (in particular to understand what is an axiomatic, and how it is related with interpretation). These questions are at the very core of Tarski's methodology of deductive science [Tarski, 1936] that permits a genuine articulation between form and content, allowing to take into account the powerful methods provided by syntax, without giving up to the advantages of the semantic approach [Sinaceur, 1991]. Geometry and figures play a special role in the teaching of proofs: the figure and its declination as interactive experiment using a dynamic geometry system questions the need for a proof for a pupil, the figure is a depiction of the semantics of the statement. Diagrammatic inferences play a crucial role in teaching proof. The teacher claims to do without it, but as we will see this is not the case in practice. Diagrammatic inferences both question the difference between syntax and semantics, and represent an alteration of the didactic contract. Diagrams are pieces of syntax which enjoy some properties of their semantics. For example, a symmetric relation is often depicted by a symmetric symbol.

1.2 The formal proof that the sum of angles of a triangle is two right angles

In this section, we describe the formal proof within the Coq proof assistant that the sum of angles of a triangle is two right angles. To describe the formal proof we need a precise context: an axiomatic setting and some definitions. The proof we describe can be formalized in the context of what Hartshorne [Hartshorne, 2000] calls an arbitrary Hilbert plane: any model of the first three groups of Hilbert axioms or equivalently Tarski's axioms $A_1 - A_9$ as listed in [Schwabhäuser et al., 1983]. This set of axioms describe the results which are valid in both hyperbolic and Euclidean geometry without assuming any continuity axiom. The plane can be non-Archimedean. We also assume in this chapter the postulate of alternate interior angles, stating that if two line are parallel the alternate interior angle of any secant are congruent. This postulate is equivalent to Euclid 5th postulate [Boutry et al., 2017].

To define the concept of sum of angles of a triangle within a computer, we could define the measure of an angle as a real and use the sum of the reals to define the sum of the angles as the sum of the measures. This is the most common approach in high-school. But formally, in a synthetic geometry setting, to define the measure of an angle, the Archimedes postulate is needed, or one need to assume the protractor postulate. In a formal setting, it is interesting as a kind of exercise in reverse mathematics to identify the minimum assumptions needed for the proofs. Therefore we chose in the library about foundations of geometry in Coq (GeoCoq) to provide a purely geometric definition of the sum of angles which make sense even in a non-Archimedean geometry and without any continuity assumption. More details about the definition of the sum of angles can be found in [Gries et al., 2016].

Instead of proving that the sum of angles is 180° we prove (as in the proof by Pythagoras above) that it is congruent to a flat angle or equivalently to two-right angles.

As noted by modern commentators of Euclid's Elements, the proofs of Euclid lack the justification for the relative position of the points on the figure. Euclid does not even provide the axioms for justifying these kind of reasoning. However, Avigad *et. al.* [Avigad et al., 2009] claim that these gaps can be filled by some automatic procedure, justifying in some sense the gaps in Euclid's original proofs.

The usual proof that the sum of angles of a triangle is two right, such as the one given by A. Amiot according to French Wikipedia[3], contains the same kind of gap. It does not provide the proof that the angles are alternate-interior angles, it is stated without proof. In this section, we give a rigorous proof, which is a translation in natural language and simplification of the formal proof which can be found in GeoCoq[4].

In formal development, we always try to prove the most generic results, that is why in the following we assume that triangles are not necessarily non degenerate and for quadrilaterals as well.

To detail the proof we need a definition of alternate-interior angles. In GeoCoq, we do not have an explicit definition of this concept[5]. But we have a definition to state that two points are on opposite sides of a line. Following Tarski, we say that the points P and Q are on opposite sides of line AB, if there is a point I which lies both on segment PQ and on line AB.

The following property is equivalent to the parallel postulate[6]:

Definition 1 (Alternate interior angles postulate) If B and D are on opposite sides of line AC and line AB is parallel to line CD then the angles $\angle BAC$ and $\angle DCA$ are congruent.

In Coq's syntax we have:

```
Definition alternate_interior_angles_postulate :=
 forall A B C D, TS A C B D -> Par A B C D -> CongA B A C D C A.
```

`TS A C B D` means that B and D are on opposite sides of line AC. `Par A B C D` means that the line AB is parallel (or equal) to line CD. `CongA B A C D C A` means that the angle BAC is congruent to angle DCA.

To obtain the formal proof we need two propositions about the relative position of point with regard to a line.

Lemma 1 *If A and C are on opposite side of line PQ, and A and B are on the same side of line PQ then B and C are on opposite side of line PQ*

[3] The comment in French Wikipedia about Amiot's proof seems to say that the proof is valid only in Euclidean geometry because it use the construction of THE parallel to line AC trough B. To be precise, the proof does not rely on the uniqueness of this line only on its existence, so this first step of the proof is valid also in hyperbolic geometry (but not in elliptic geometry). The Wikipedia comment fails to notice that essential use of a version of the parallel postulate relies in the use of what we called above the postulate of alternate-interior angles.

[4] `http://geocoq.github.io/GeoCoq/html/GeoCoq.Meta_theory.Parallel_postulates.alternate_interior_angles_triangle.html#`

[5] We may add a definition of alternate-interior angles, which would be a short cut for the predicate TS which states that two points are on opposite sides of a line, but adding more definitions make the formal proofs more cumbersome, that is why we hesitate to introduce a new definition.

[6] Note that the reciprocal is valid in neutral geometry

In Coq's syntax, this lemma (which is present in the ninth chapter of [Schwabhäuser et al., 1983]) is stated as:

```
Lemma l9_8_2 : forall P Q A B C,
  TS P Q A C -> OS P Q A B -> TS P Q B C.
```

`OS P Q A B` means that A and B are on the same side of line PQ.

We also need the following lemma which is not present in [Schwabhäuser et al., 1983]:

Lemma 2 *If Y and Z are on the same side of line AX, and X and Z are one opposite sides of line AY then X and Y are on the same side of line AZ.*

```
Lemma os_ts1324__os : forall A X Y Z,
  OS A X Y Z -> TS A Y X Z -> OS A Z X Y.
```

We have now all the properties required to prove the main theorem:

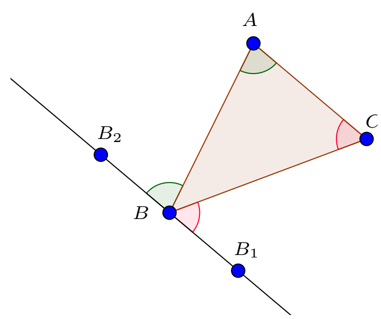


Fig. 1: The sum of angles of a triangle.

Theorem 1 *Assuming the postulate of alternate interior angles, the sum of angles of any triangle is congruent to the flat angle.*

Proof Let ABC be a triangle, we need to show that the sum of angles is the flat angle. If the points ABC are collinear then the sum of angles is a flat angle[7]. Let l be a parallel to line AC through B[8] (see Fig. 1). Let B_1 be a point on the line l such that B_1 is on the opposite side of A with regard to the line BC. Let B_2 be the symmetric of B_1 through B[9].

[7] We have a separate lemma for this case, we could also assume that we have a proper triangle. In formal development, we always try to prove the most generic results.

[8] Note that we do need "the parallel line", uniqueness is not important here

[9] We could also use any point B_2 such that B belongs to segment B_1B_2.

> C and B_2 are on opposite sides of line AB because:[a]
> B_1 and B_2 are on opposite sides of line AB (*) because by construction segment B_1B_2 intersects line AB in B.
> As AC is strictly parallel to line B_1B_2, A and C are on the same side of line B_1B_2.
> By Lemma 2, we have that C and B_1 are on the same side of line AB.
> Hence, using Lemma 1 and fact (*), we can conclude that C and B_2 are on opposite sides of line AB.
>
> [a] Maybe there is a simpler proof ? but for sure we need to use the fact that AC is parallel to B_1B_2

By the construction of B_2, B_1BB_2 is a flat angle, hence it suffice to show that the sum of angles is congruent to the angle B_1BB_2.

By the postulate of alternate interior angles, we have that the angle $\angle ABB_2$ is congruent to $\angle CAB$. By construction, the angles $\angle CBB_1$ and $\angle BCA$ are alternate, hence by the postulate of alternate interior angles as the lines BB_1 and CA are parallel, the angle $\angle CBB_1$ is congruent to $\angle BCA$. □

We give now a slightly different version of the proof, assuming that angles can be copied on a given side of a line (this is an axiom in Hilbert's foundations of geometry).

Proof Let B_1 be a point on the opposite side of A with regard to the line BC such that the angle $\angle ACB$ is congruent to the angle $\angle CBB_1$. As the alternate angle are congruent, the line AC is parallel to line BB_1. Let B_2 be a point on the line BB_1 such that B belongs to segment B_1B_2. ... □

The rest of the proof is the same as in the first version.

1.3 Some questions and issues raised by the formalization of the proof in Coq

The formal proof differs from the proof which is taught in high-school, because using a proof assistant, all steps of the proof have to be justified. The proof assistant prevents us from deducing facts from the figure, this reflects the didactic contract between the teacher and the pupil. It is interesting to distinguish in the formal proof, intermediate steps which can be considered as uninteresting details from the steps which can be considered as proper gaps in the informal proof.

We think that the part of the proof which is typeset in a frame can not be considered as an uninteresting detail, it is an important sub-statement, whose justification is not obvious.

From a didactic point of view, there are steps of the proof that should remain *implicit* in a classroom and steps that should be emphasized.

This choice should be made consciously and depending on the context. For example, the transitivity of parallelism can either be explicit or implicit depending on the curriculum/class of the student. The famous example of the fallacious proof that all triangle are isosceles shows that the relative position of the points should not always be taken for granted, but the example of the sum of angles of a triangle is maybe too subtle to be studied rigorously in high-school. In the example studied in this section, we believe that in a classroom it should be stated explicitly that the fact the angles are alternate is assumed.

For an integration of this exercise in an AI milieu, the tool-box would display different theorems/construction tools[10]: at least the postulate of alternate-interior angles and the tool to construct parallel lines. Should we have a tool to construct a point on a line on the opposite side of a point ? We see here the impact of the AI milieu on the didactic setting[11].

For an integration in an AI milieu, we would need to automate some steps of the proof which are purely administrative burden, at least the ones which are present in the original Coq code (see appendix) and that we kept implicit in this chapter. It includes: using the fact that the sum of angles is a morphism with regard to congruence of angles ($\alpha \equiv \alpha' \Rightarrow \alpha + \beta = \alpha' + \beta$), the fact that the sum of angles is unique up to congruence, and various permutation properties of the manipulated predicates,…

2 Varignon's theorem

In this section, we will provide various proofs of Varignon's theorem that can be provided with the knowledge developed in the French secondary curriculum or at the beginning of university, that we will analyse with the question raised in Section 1.1.

Varignon's theorem, states that:

Theorem 2 *Let $ABCD$ be a quadrilateral. Let I, J, K and L be the midpoints of AB, BC, CD, and AD, then $IJKL$ is a parallelogram.*

[10] Construction tools correspond to existence theorems.

[11] The proof could also be modified to construct B_1 and B_2 such that B belongs to segment B_1B_2 and then say that at least one of them is on the opposite side of A with regard to the line BC.

2.1 Logical analysis of a classical proof of Varignon's theorem

The usual proof presented in classroom is based on the midpoint theorem as the original proof (See[12] Fig.2) but this proof suffers from one problem, as we discuss below.

Si les côtés *AB*, *BC*, *CD*, *DA* d'une figure rectiligne de quatre côtés, ſont diviſés chacun en deux parties égales en *F*, *G*, *H*, *E*, & que les points des diviſions ſoient joints par les lignes droites *FE*, *EH*, *HG*, *GF*, la figure quadrilaterale *FEHG* eſt un parallélograme; car en menant les lignes *DB*, & *AC*, comme par l'hypothéſe, *AF* = *FB* & *AE* = *ED*, *AF*· *FB* :: *AE*· *ED*, & ainſi (*Part.* 2.) *EF* eſt paralléle à *DB*. De même puiſque, par l'hypothéſe, *BG* = *GC*, & *DH* = *HC*; *BG*· *GC* :: *DH*· *HC*, & par conſéquent (*Part.* 2.) *GH* ſera encore paralléle à la ligne *BD*. Donc *EF* & *GH* ſont paralléles à la même troiſiéme ligne, elles ſont donc auſſi paralléles entre elles. On peut par la même raiſon prouver que les lignes *FG* & *EH* ſont paralléles à la ligne droite *AC*, & par conſéquent paralléles entre elles. Donc le quadrilatére *EFGH* eſt un parallélograme.	If the sides AB, BC, CD, DA of a quadrilateral are each divided into equal parts in F, G, H, E, and the division points are joined by lines FE, EH, HG, GF the quadrilateral $FEHG$ is a parallelogram; because by drawing the lines DB and AC, as by hypothesis, $AF = FB$ and $AE = ED$, $AF.FB :: AE.ED$, and hence (Part 2) EF is parallel to DB. Similarly, by hypothesis $BG = GC$ and $DH = HC$; $BG.GC :: DH.HC$ and hence GH is also parallel to line BD. Hence, EF and GH are parallel to the same line and therefore are parallel. Similarly, it can be proved that line FG and EH are both parallel to line AC and therefore are parallel. Hence the quadrilateral $EFGH$ is a parallelogram.

Fig. 2: Original proof of Varignon's theorem and English translation

The standard proof is the following:

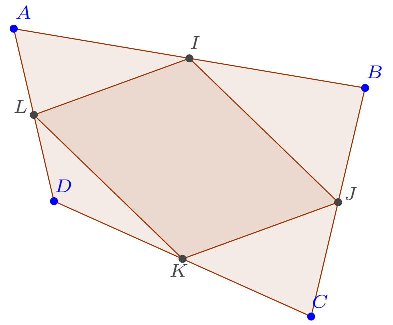
Proof Consider triangle ABC, by the midpoint theorem we have that AC is parallel to IJ. Using again the midpoint theorem in triangle ACD we have that LK is parallel to AC. Hence by transitivity of parallelism, we have that IJ is parallel to LK. Similarly, we have that IL is parallel to JK. Hence, $IJKL$ is a parallelogram. □

A variant of the proof consists in using the characterization of a parallelogram as a quadrilateral with a pair of opposite sides which are parallel, congruent, and whose diagonals intersect.

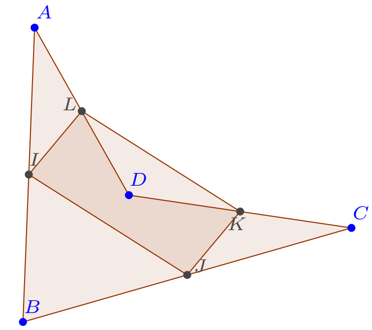
2.2 Issues and challenges raised by the formalisation of the classical proof of Varignon's theorem

The problem with this proof is at the last step, the theorem which says that if the opposite side of quadrilateral are parallel then it is a parallelogram requires that the parallelism is strict i.e. the lines do not coincide. But, it could be the case that the points I, J, K and L are on the same line as shown on Figure 3d. So in the formal version of *this* proof, we need to add the fact that I, J and K are not collinear. This restriction is not welcome

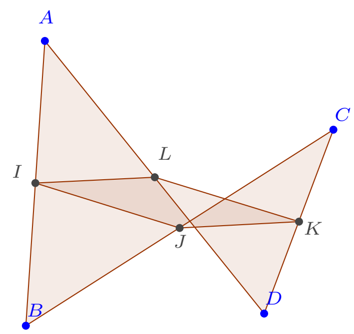
[12] `http://polib.univ-lille3.fr/documents/B590092101_000000011.489_IMT.pdf`



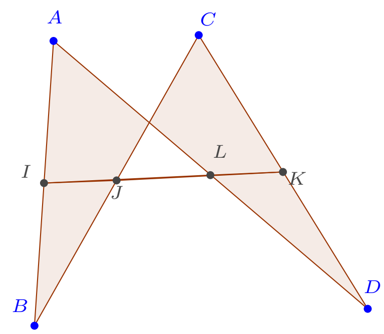
(a) Convex case



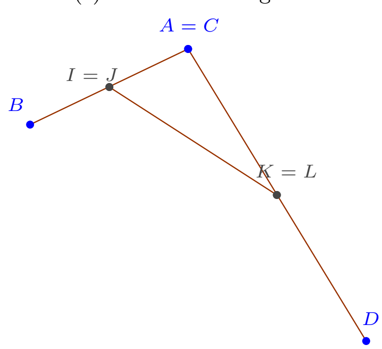
(b) Concave case



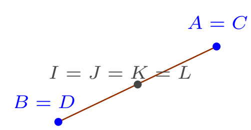
(c) Self-intersecting case



(d) Special case



(e) Case $A = C$



(f) Case $A = C$ and $B = D$

Fig. 3: Varignon's theorem

because even in the case where I , J and K are on the same line, then $IJKL$ is a parallelogram in the sense that its diagonals meet in their midpoints, and the opposite side are congruent (in GeoCoq's formalization we call this figure a flat parallelogram). The formal proof also differs from the informal proof because we assume explicitly that $A \neq C$ and $B \neq D$ to ensure that the sides of $IJKL$ are proper lines.

2.3 Logical analysis of alternative proofs of Varignon's theorem

Beside the classical proof of Varignon theorem in synthetic geometry, there are other mathematical settings that allow proving this theorem. We provide below some examples.

2.3.1 A proof using vectors

Proofs in this setting rely on the vector characterisation of a parallelogram. We assume here just the existence of four points without any hypothesis. It is necessary to introduce vectors that correspond to ordered pair of points. Each ordered pair of points determines a vector, such that for a given parallelogram, there are potentially 12 non-zero vectors. Providing the characterization, as given below, necessitates to identify that only pairs of consecutive points are relevant, and that the two pairs should be in opposite order compared to the initial order of the four points.

Given four points M, N, P, Q, $MNPQ$ is a parallelogram if and only if $\overrightarrow{MN} = \overrightarrow{QP}$ (resp. $\overrightarrow{MQ} = \overrightarrow{NP}$)

In this mathematical setting, the Varignon's theorem can be reformulated as:

Given four point A, B, C and D and I, J, K and L the midpoints of the segments AB, BC, CD and AD, $\overrightarrow{IJ} = \overrightarrow{KL}$ (resp. $\overrightarrow{IL} = \overrightarrow{JK}$)

Proof 2

Let A, B, C and D be four points in the plane, and I, J, K and L the midpoints of the segments AB, BC, CD and AD. Prove that $\overrightarrow{IJ} = \overrightarrow{KL}$ (resp. $\overrightarrow{IL} = \overrightarrow{JK}$).

$\overrightarrow{IJ} = \overrightarrow{IB} + \overrightarrow{BJ}$ (Vector addition) (1)

$\overrightarrow{IB} = \frac{1}{2}\overrightarrow{AB}$; $\overrightarrow{BJ} = \frac{1}{2}\overrightarrow{BC}$ (Vector characterisation of the midpoint of a segment) (2)

$\overrightarrow{IJ} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$ (Substitution) (3)

$\overrightarrow{IJ} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC})$ (Factorisation) (4)

$\overrightarrow{IJ} = \frac{1}{2}\overrightarrow{AC}$ (Vector addition) (1) (5)
$\overrightarrow{LK} = \overrightarrow{LD} + \overrightarrow{DK} = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{DC}) = \frac{1}{2}\overrightarrow{AC}$ (6)

From (3) and (6), we conclude that $\overrightarrow{IJ} = \overrightarrow{LK}$ (transitivity of equality relation)

There are several keys in this proof. The first one is to decompose $\overrightarrow{IJ}$ thanks to vector addition, using two vectors, one with end B and the other with origin B. A second key is to use the characterization of a midpoint of a segment as an equality between vectors, and then to perform substitution and factorization, which are more general actions, present in AI. Formalizing such proof will require to explicit the choice to be done along the proof. It is possible to do these transformations without referring to a geometrical drawing, but it seems rather clear that having a drawing of a generic convex quadrilateral, even a freehand drawing is a powerful support for choosing the adequate transformation. Nevertheless, the particular cases do not need to be made explicit, because the vector characterisation of a parallelogram, that we have recalled above, does not need any non degeneracy conditions. The conclusion relies on the transitivity of equality. It is noticeable that this property of equality shapes the proof that we provided. Another way of doing would be to continue from step (5) by introducing point D to decompose AC: $\overrightarrow{IJ} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{DC}) = \frac{1}{2}(2\overrightarrow{LD} + 2\overrightarrow{DK}) = \frac{1}{2}(2(\overrightarrow{LD} + \overrightarrow{DK})) = \overrightarrow{LK}$. We could hypothesize that formalizing such proofs will need to make explicit the way of choosing how to implement the successive transformations.

2.3.2 Two proofs using Cartesian coordinates

Proof 3

This proof relies on the characterisation of parallelogram as quadrilaterals whose diagonals intersect at their common midpoint, and on the characterisation of the midpoint of a segment by the mean of coordinates. The goal is to prove that segments IK and JL have the same midpoint. In this proof, once the four points have been introduced, the first thing to do is to choose three points that will serve as reference for determining the coordinates. In order to make calculations easier, we chose these three points among the four given ones. For example, let us choose A, B and C with coordinates $A:(0,0)$, $B:(0,1)$ and $C:(1,0)$; then D has undetermined coordinates (a,b). Using the fact that the coordinates of the midpoint of a segment are the half of the sum of the coordinates of each point, we get the following:

The coordinates of I, J, K and L are respectively $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(\frac{a+1}{2}, \frac{b}{2})$ and $(\frac{a}{2}, \frac{b+1}{2})$.

Let O_1 be the midpoint of segment IK. The coordinates of O_1 are $(\frac{a+1}{4}, \frac{b+1}{4})$ (1). Let O_2 be the midpoint of segment JL. The coordinates of O_2 are $(\frac{a+1}{4}, \frac{b+1}{4})$ (2). From (1) and (2) we conclude that $O_1 = O_2$ (transitivity of equality) As a consequence $IJKL$ is a parallelogram (characterisation of parallelograms using common midpoint of the diagonals).

Remark 1: The choice of three generic points as reference leads to accessible calculations. In the perspective of formalization, it has the advantage of not introducing di-symmetry between the four initial points.

Remark 2: For this proof, once decided to use Cartesian coordinates, the way of doing is rather systematic. In this respect, we may hypothesize that the formalisation will fit more the pen/papers proof than it was the case for Proof 2.

Proof 3*

This proof relies on the same characterization by the common midpoint of the diagonals, but using lines equations. The aim is to prove that the two lines IJ and KL intersect at the midpoints of the segments IJ and KL. This necessitates introducing the two lines, their equation, to solve the system of equations and to check that the ordered pair of solutions of the equation is identical to the midpoint of both segments. This method is congruent with the classical way of characterising a parallelogram as a quadrilateral whose diagonals intersect at their common midpoint, however the calculations are less easy than in proof 3.

Proof 4

This proof relies of the metric characterisation of a parallelogram as a quadrilateral with opposite sides having same length. The first step in this case is to choose an origin and two *orthogonal* axes, in order to be able to use the fact that the distance in a Euclidean plane is the square root of the sum of squares of the coordinates differences. In order to make calculations easier, we choose A as origin, and AB with B $(0, 1)$ as first axis. Then, the coordinates of C and D are indeterminate ones. Let them be (c_x, c_y) for C and (d_x, d_y) for D. Then the coordinates of the midpoints are $I : (0, \frac{1}{2}); J : (\frac{c_x}{2}, \frac{1+c_y}{2}); K : (\frac{c_x+d_x}{2}, \frac{c_y+d_y}{2}); L : (\frac{d_x}{2}, \frac{d_y}{2})$

Then using the formulas recalled above:

$IJ = \sqrt{((c_x/2)^2 + (c_y/2)^2}$ (1)

$LK = \sqrt{((c_x/2)^2 + (c_y/2)^2}$ (2)

$JK = \sqrt{((d_x/2)^2 + ((d_y-1)/2)^2}$ (3)

$LK = \sqrt{((d_x/2)^2 + ((d_y-1)/2)^2}$ (4)

From (1) and (2) we conclude that $IJ = LK$, and from (3) and (4) we conclude than $JK = KL$. Hence, as the quadrilateral $IJKL$ has his opposite sides with same lengths, we conclude that it is a parallelogram. Both remarks done on proof 3 above holds for this proof.

2.3.3 About formalization of the alternative proofs

The formalization of **Proof 3** highlights again some necessary non degeneracy conditions. Indeed the characterization of parallelograms using midpoints is called `mid_plg` in GeoCoq:

```
mid_plg : A <> C \/ B <> D ->
        Midpoint M A C ->
        Midpoint M B D -> Parallelogram A B C D
```

To use it, we need to prove that either I is different from K or J is different from L. Assuming that A is different from B, C and D, the disjunction can be proved by contradiction. If we had $I = K$ and $J = L$ then both $ACBD$ and $ACDB$ would be parallelograms, which is impossible.

To formalize the analytic proof, there are two solutions: either we consider that the geometric objects are **defined** by the algebraic equations, but then the proof is not about geometry, it is about algebra. It would be possible to prove that the geometric predicates defined by the algebraic equations over their coordinates verify the usual geometric axioms. But this would only prove that $\mathbb{R}^2$ is a model of the axioms. To fully justify the analytic method, it is necessary to prove that the algebraic computations can be performed geometrically, following Descartes [Descartes, 1925]. This proof is called the arithmetization of geometry. This is the culminating result of both Hilbert's *Foundations of Geometry* [Hilbert, 1960] and Schwabhäuser, Szmielew and Tarski's book [Schwabhäuser et al., 1983]. An analytic proof in geometry, can be seen as a geometric proof, thanks to a meta-theoretical argument: both theories have the same models. Hence, for the formalization of the analytic proofs, starting from a synthetic axiomatic setting, we rely on the Coq formalization of the arithmetization of geometry [Boutry et al., 2019]. Technically it means that each arithmetic operation is a shortcut for a geometry construction. Having prove that these operation form a field, then we can forget about the geometry and resort to computations. From a didactic point of view, this raises the question how this link between two different perspective could be presented to present a proof which is both accessible and rigorous. Sometimes, in educational context a bijection between the real line and the geometric line is assumed. In this particular example, and many other geometric statements, we do not need the reals, the formalization shows that the Cartesian plane over a Pythagorean field is sufficient.

In GeoCoq, the change of perspective from synthetic geometry to analytic geometry, can either be performed manually (by using the characterisation of midpoint using coordinates) or automatically using a tactic called `convert_to_algebra`. In some sense, the formal proof, as tactic, provides an explanation of the method used to find the proofs. The availability of automation within interactive prover blurs the lines between proofs as explicit objects, and programs generating proofs.

Note that the proving geometric statements using analytic means can lead to some kind of circular proof, if the geometric statement is used in the proof of the arithmetization of geometry. This is the case of the Pythagorean theorem: an analytic proof is straightforward, but the Pythagorean theorem is needed in the proof of the arithmetization of geometry, specifically in the characterization using coordinates of the congruence of segments. Our formal proof of Pythagoras' theorem itself employs the intercept theorem (also known in France as Thales' theorem) so an analytic proof of the intercept theorem would also be somewhat circular.

The analytic proofs by substitution can also either be performed manually by manipulating algebraic equations, or automatically using Gröbner basis algorithm or Gauss elimination. This approach could lead in the future to new presentation of proofs at the frontier between maths and computer science, by making explicit and systematic the heuristic used by the student and turning them in some cases into proper algorithms. Automated theorem proving could play the same role with regards to proof that computer algebra system with regards to computations.

For the formalization of **Proof 4**, the choice of the coordinate axis requires some reasoning. It relies on the invariance of geometric predicates by translation and rotation, so that one can assume simple coordinates for A,B and C, see Section 3 of [Genevaux et al., 2011] for an example or [Harrison, 2009]. For this proof, once again the non degeneracy conditions complicate the proof. Indeed one should pay attention that for proving that the quadrilateral is a parallelogram using the fact that opposite side have the same length, one needs to prove that the quadrilateral is not crossed, in the sense that the diagonals have a common point, and the figure should be either non flat, or fully flat:

```
Lemma cong_cong_parallelogram:
      forall A B C D P : Tpoint,
       Cong A B C D ->
       Cong B C D A ->
       (~ Col A B C \/ (Col A B C /\ Col A B D)) ->
       B <> D ->
       Col A P C ->
       Col B P D -> Parallelogram A B C D.
```

2.3.4 Alternative proofs using automated deduction

Area method

In this subsection, we give as example the proof of Varignon's theorem using the area method. The area method is a procedure for a fragment of Euclidean plane geometry [Chou et al., 1994, Zhang et al., 1995, Janičić et al., 2012]. It is based on the concept of signed area of triangles, and can efficiently prove many non-trivial theorems and produces proofs that are often very concise. This method has been taught to student preparing mathematics Olympiad in China. For a quick overview of the method see [Narboux et al., 2018]. Using the signed area, a number of geometric predicates can be simply expressed, for instance: Three points are collinear iff the signed area of the triangle is zero:

Col $A\,B\,C$ iff $\mathcal{S}_{ABC} = 0$;

Two lines are parallel, if they are at constant distances, and we can characterize it using signed area: $AB \parallel CD$ iff $A \neq B \wedge C \neq D \wedge \mathcal{S}_{ACD} = \mathcal{S}_{BCD}$, etc.

The method consists in eliminating the points one by one using formulas based on the way the point is constructed. In our example, we only have one construction: the midpoint and we use the following lemma about signed areas: if I is the midpoint of AB then for any P and Q: $\mathcal{S}_{PQI} = \frac{\mathcal{S}_{PQA}}{2} + \frac{\mathcal{S}_{PQB}}{2}$

Then we use the characterization of a parallelogram as a quadrilateral whose opposite sides are parallel. We give here the proof that two sides are parallel. Formally, to prove that it is a parallelogram this piece of proof should be duplicated or an argument of symmetry should be used. We need to show that $IJ \parallel LK$ this is equivalent to $\mathcal{S}_{KIJ} = \mathcal{S}_{LIJ}$.

$$\begin{array}{ll}
\mathcal{S}_{KIJ} - \mathcal{S}_{LIJ} & \\
= \frac{\mathcal{S}_{KIB}}{2} + \frac{\mathcal{S}_{KIC}}{2} - \frac{\mathcal{S}_{LIB}}{2} - \frac{\mathcal{S}_{LIC}}{2} & J \text{ Eliminated} \\
= \frac{\mathcal{S}_{BKA}}{2} + \frac{\mathcal{S}_{BKB}}{2} + \frac{\mathcal{S}_{CKA}}{2} + \frac{\mathcal{S}_{CKB}}{2} - \frac{\mathcal{S}_{BLA}}{2} - \frac{\mathcal{S}_{BLB}}{2} - \frac{\mathcal{S}_{CLA}}{2} - \frac{\mathcal{S}_{CLB}}{2} & I \text{ Eliminated} \\
= \frac{1}{2}(\mathcal{S}_{BKA} + \mathcal{S}_{CKA} + \mathcal{S}_{CKB} - \mathcal{S}_{BLA} - \mathcal{S}_{CLA} - \mathcal{S}_{CLB}) & \text{Simplification} \\
= \frac{1}{2}(\frac{\mathcal{S}_{ABC}}{2} + \frac{\mathcal{S}_{ABD}}{2} + \frac{\mathcal{S}_{ACC}}{2} + \frac{\mathcal{S}_{ACD}}{2} + \frac{\mathcal{S}_{BCC}}{2} + \frac{\mathcal{S}_{BCD}}{2} - \mathcal{S}_{BLA} - \mathcal{S}_{CLA} - \mathcal{S}_{CLB}) & K \text{ Eliminated} \\
= \frac{1}{2}(\frac{\mathcal{S}_{ABC}}{2} + \frac{\mathcal{S}_{ABD}}{2} + \frac{\mathcal{S}_{ACC}}{2} + \frac{\mathcal{S}_{ACD}}{2} + \frac{\mathcal{S}_{BCC}}{2} + \frac{\mathcal{S}_{BCD}}{2} - \frac{\mathcal{S}_{ABA}}{2} - \frac{\mathcal{S}_{ABD}}{2} - \frac{\mathcal{S}_{ACA}}{2} - \frac{\mathcal{S}_{ACD}}{2} - \frac{\mathcal{S}_{BCA}}{2} - \frac{\mathcal{S}_{BCD}}{2}) & L \text{ Eliminated} \\
= \frac{1}{4}(\mathcal{S}_{ABC} + \mathcal{S}_{BCA}) & \text{Simplification} \\
= 0 & \text{Simplification}
\end{array}$$

This proof can be obtained automatically.

The same method can be applied to obtain systematically a proof using Cartesian coordinates. As the midpoint relation can be expressed as a linear equality, the goal can be solved by a simple Gauss elimination algorithm (for more involved theorems, one would need to use the method of Wu or Gröbner basis which are also available in proof assistants [Genevaux et al., 2011, Pottier, 2008]). The choice of the best characterization of parallelograms for the computational proof (the diagonals intersect in their midpoint) is still to be decided by a human in the current implementation. The AI milieu of GeoCoq is up to our knowledge the unique setting where synthetic and analytic reasoning can be intermixed and validated formally.

2.4 Didactical implication

The recognition by advanced mathematics students that a given theorem can be stated and proved in a variety of mathematical settings is an important issue. Indeed, as we can experience as university teacher, it is often the case that students think that there is exactly one proof for a given theorem. This is particularly important for prospective secondary mathematics teachers who should be able to recognize the underlying theory of the didactical transposition's choices made in secondary curriculum. This resort of the so-called second discontinuity of Klein that "concerns those who wish to return to school as teachers and the (difficult) transfer of academic knowledge gained at university to relevant knowledge for a teacher" [Winsløw and Grønbæk, 2014]. For this purpose, the formalization in Coq raises new issues and opportunities. A first issue has already been mentioned in the first example; it concern the need for considering the question of degeneracy that is often hidden in pencil/paper proofs in Geometry, in particular because it is usual that drawings support our intuition and orient the proofs while "it is clear that some evidences in the drawing register have to be questioned in the theory" [Durand-Guerrier and Arsac, 2009]. As shown in Sec. 2.3, this is crucial in the case of Varignon's theorem. A second issue concerns the importance of making explicit the role of axiomatic theories in the proving process, that as stressed by [Planchon and Hausberger, 2020, p.162], is not easily recognized by students involved in a prospective mathematics teacher training program. For formalizing geometrical proofs in Coq, one needs to refers explicitly to the assumed axiomatic and open the possibility of discussing it with students, as a contribution to make them aware of the implicit assumptions they have developed during their academic studies. It also opens the possibility of discussing if and why different theories are interpreted by the same models. An important example is the case of synthetic geometry and analytic geometry. This issue is nearly never discussed in undergraduate mathematics studies, while it could offer sound justifications for the omnipresent practice in high school of moving from synthetic geometry to analytical geometry and vice versa. Another example which is not always clearly stated in the undergraduate studies, is the justification of the use of complex numbers in geometry.

In the following section, we present a teacher training session on Varignon's theorem that has been implemented for years by the second author in a pencil/paper modality and we discuss the relevance of introducing formal proof in the milieu of the situation to test the relevance of the link between pencil/paper and formal proofs.

3 A teacher training session on Varignon's theorem

N.B. This is based on the implementation of this session in a module for future mathematics teachers entitled Didactics and epistemology of mathematics in Master 1 at the University of Montpellier. This module has not been studied in the context of research. This is a perspective to test the relevance of the link between pencil/paper and formal proofs.

3.1 Context, motivation, description and a priori analysis of the session

The work around Varignon's theorem that we present below has been implemented during several years in a module of Didactic and Epistemology of Mathematics in the first year of the master degree for prospective mathematics teachers ("Metiers de l'enseignement et de la formation") in France in Montpellier, where 10 hours where dedicated to proofs. Motivations of the introduction of a dedicated work on proofs in secondary mathematics teacher training are double-faced. On the one hand, it is necessary to provide prospective teachers with proofs knowledge and skills, that is in general not sufficient to address the professional needs despite the fact that they have practice proofs in their academic studies. On the other hand, as stressed by [Durand-Guerrier and Tanguay, 2018], we hypothesize that "[...] working with proof is likely to contribute to conceptualization by prompting a work with the mathematical objects at stake, in agreement with the syntax-semantics dialectic in proof and proving (e.g. [Weber and Alcock, 2004])." (op. cit, p. 20)

The choice of the Varignon's theorem relies on its potential to deal with both aspects mentioned above. For example, [Durand-Guerrier et al., 2012] considered that Varignon's Theorems : " [provide] a situation relevant to secondary school and teacher education, in which a given definition must be explored in order to identify the entire range of objects that satisfied it" (op. cit. p.380)

3.1.1 Production and analysis of Varignon's Theorem proofs

In the first part of the activity, students were invited to provide two proofs of Varignon's theorem in two different mathematical settings. The theorem is given as above: "Given any quadrilateral, the respective midpoints of its sides are the vertices of a parallelogram." Once this done, the students were invited to analyse their proofs from the point of view of objects introduced, properties or relations mobilized, theorems used and methods of reasoning implemented. They were also required to precise in which respect generality was addressed in their proof. In some cases, they worked in pair for the analysis, while in other cases they worked individually for both steps. A main goal of this first part of the activity was to push students to go beyond the classical geometrical proof based on the midpoint theorem, and to engage them in a logical analysis of the proofs in different mathematical settings with a focus on objects, properties, relations on the one hand, theorems and modes of reasoning on the other hand.

3.1.2 Exploring the inverse problem

In this part of the activity, students are asked to explore the three inverse problems below: Which necessary and sufficient condition should be satisfied by the initial quadrilateral in order to obtain:

1. a rhombus
2. a rectangle
3. a square

The general idea is here that the property of the sides of the parallelogram comes from the property of the diagonals of the initial quadrilateral. A rhombus is a parallelogram with four isometric sides. As a consequence, in order to get a rhombus, the diagonals of the initial quadrilateral should be isometric. This condition is necessary. It is also sufficient. A rectangle is a parallelogram with perpendicular adjacent sides. As a consequence, in order to get a rectangle, the diagonals of the initial quadrilateral should be perpendicular. This condition is necessary. It is also sufficient. A square is both a rhombus and a rectangle. As a consequence, in order to get a square, it is necessary and sufficient that the diagonal of the initial quadrilateral are isometric and perpendicular. An implicit hypothesis might be that the expected condition be expressed in term of "type of quadrilateral". This would lead to answer with sufficient conditions, such as "starting from a rectangle provides a rhombus", "starting from rhombus provides a rectangle" and "starting from a square provides a square". Questioning the necessity should lead to move to the property of diagonals. Nevertheless, we cannot exclude that the condition be considered as necessary by some students. Indeed, in the French curriculum, characterizing quadrilaterals by the property of their diagonals is considered in general only for parallelograms, with an exception with kites. As [Durand-Guerrier, 2003] stresses, 60 of 273 students just entering a French university answered that "a quadrilateral with perpendicular diagonals" is a rhombus, some of them adding that the only counterexample they know is the square, but it is a particular rhombus [Durand-Guerrier, 2003, p. 26].

3.2 Account of naturalist observation along several years

The classical proof is in general the first one that is produced ; the alternative proofs that we have presented above are regularly provided by students, in some cases with redundancy : for example, for the proof using vectors, some students prove the equality of two pairs of vectors instead of one pair. In the classical proof, the midpoint theorem and the inference rule at stake (*modus ponens*) is generally explicit; however other theorems are used without being mentioned. Below is an example of the analysis of the classical proof by a student[13] (E5) :

> "It is a direct proof starting from a generic quadrilateral and the midpoints of the sides and implying properties until the conclusion of the proof that validate the theorem. The midpoint theorem [not stated] and the definition of a parallelogram are used in the proof. Also the property, implicit here, saying that if two lines are parallel to a same third line, then they are parallel to each other are used here. Of course, the definition of a quadrilateral is implicit. For this proof, we have introduced the name of the points. The generality is taken in account here as we do not give other precision and the initial quadrilateral that remain generic, except the name of the vertices."

Concerning the proofs using vectors or Cartesian coordinates, there are still more implicit assumptions. Indeed, such proofs rely mainly on computation, and in this case, theorems and inferences remain in general implicit. For example, student E6 provided a proof using vectors, and in his analysis he wrote "No theorem is used". Another student (E8) provided a proof using Cartesian coordinates to prove the equality of a pair of vectors. In her analysis, she wrote

> "There is no use of theorem, but rather we use the definition of vector equality. The reasoning is an algebraic one in Cartesian geometry."

While keeping implicit some elements in a proof might not be problematic for experts, we hypothesize that it is important for prospective teachers to identify such implicit steps in their own proof. Indeed, there is research evidence that they might not be shared or recognized by a number of secondary students (e.g. implicit quantification on conditional statements as stressed in [Durand-Guerrier, 2003]). Concerning the second part of the activity with the inverse problem, a majority of students over the years provide the sufficient condition, some of them claiming that it is necessary, and in some cases providing a "proof". Below are two examples from the same corpus as above. Student E2 wrote

> "4-a) A rhombus is a parallelogram with diagonals intersecting in their midpoint and perpendicular to each other: the initial figure is necessarily a rectangle. b) a rectangle is a parallelogram with diagonals of same length: the initial figure is necessarily a rhombus. c) a square is a parallelogram with diagonals of same length and perpendicular: the initial figure is necessarily a square."

[13] Excerpts from students' productions are translated from French.

Student E10 proved that if we get a square, the diagonals of the initial quadrilateral have same length and are perpendicular. She concludes as below:

> "4-c) Hence $ABCD$ is a quadrilateral with diagonal of same length and perpendicular, hence it is a square."

She did the same for a) and b). During the session where the answers were collected, only two students among 11 provided the conditions on the diagonals without concluding with a category of parallelogram, which was generally the case in other sessions. An interpretation is that the students reason in the domain of parallelograms, not in the domain of more general quadrilaterals. This might be an effect of the common practice of letting implicit the quantification, with a consequence that the domain of objects at stake remains also implicit, this being reinforced by the fact that the students have encountered mainly the parallelograms and the common particular ones (rhombus, rectangle, square).

3.3 Evolution of the teacher training session by introducing a proof assistant

The brief account we give of the students work is in line with the naturalistic observation done along years with this activity. We hypothesize that introducing in the activity the possibility of checking the proof with a proof assistant might help students to identify more precisely the logical structure on the one hand (e.g. recognizing the use of theorems in proofs using vectors or Cartesian coordinates), and discussing the generality in a more accurate way. We remark that using a proof assistant founded on a logical framework which allows separation of reasoning from computation such as type theory of deduction modulo [Dowek, 2014] or simply a proof language allowing the description of automatic procedure, could give sense to the assertion: "there is no use of theorem". The fully automatic proof of Varignon's theorem using the area method which can be found in the appendix is an example of such a 'theorem less' proof. The contrastive analysis of the pencil/paper proofs with the formal ones open paths for designing an adaptation of this teacher training session.

We propose the following scenario: after a familiarization with a proof assistant such as Coq, we ask for two different proofs of Varignon's theorem and to choose one and to formalize it within Coq using the GeoCoq library. Then we can discuss what the formalization brings in the analysis of the proof. Our hypothesis is that formalization will produce a finer and more rigorous analysis of the proof, making clear reference to the underlying properties or axioms which are necessary for the change of mathematical settings. Then, we propose to solve the inverse problem on pencil/paper and later check the proof using the proof assistant. It will be interesting to study the impact of this alternation between the two modalities on the clarification of the concepts involved in the proofs and on the concept of proof per se. We must acknowledge that having an audience acquainted to a proof assistant is a demanding prerequisite of the proposed scenario.

Conclusions

In this chapter, we have presented an exploratory study aiming to discuss the relationships between pencil/paper proofs and formal ones in a didactic perspective. We first discuss the classical proof that the sum of the angles of a triangle is two right angles, putting in evidence the points raised by the formalisation in a proof assistant, and discussing which of the implicit steps would be relevant to clarify in an educational setting, in particular in teacher training.

Our second example, the theorem of Varignon, has been chosen for its potentiality a priori to feed the discussion on the interaction between pencil/paper proofs and formal ones. This has been evidenced for the proof in synthetic geometry, and for the alternative proofs, the question raised being far beyond the specific example of the Varignon theorem, for example for what concerns the back and forth between synthetic and analytic geometry, which is taken for granted in secondary curriculum in France, and certainly in other educational systems.

We hypothesize that the teacher training briefly presented in section 3 will be improved by the introduction of a proof assistant in the milieu, in order 1/ to enrich the experience of prospective teachers for what concerns proofs, a professional competence that need to be developed as it is evidenced in international literature ; 2/ to improve their knowledge on the relationships between Synthetic Geometry and Analytic Geometry, thanks to the clue questions raised by formalisation.

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Appendix: Verbatim of the formal proofs

We first give the formal proof of the fact that the sum of angles of a triangle is congruent to the flat angle.

```
Section alternate_interior_angles_postulate_triangle.

Context `{T2D:Tarski_2D}.
```

```
Lemma alternate_interior__triangle :
 alternate_interior_angles_postulate -> triangle_postulate.
Proof.
  intros aia A B C D E F HTrisuma.
  elim(Col_dec A B C).
    intro; apply (col_trisuma__bet A B C); auto.
  intro HNCol.
  destruct HTrisuma as [D1 [E1 [F1 []]]].
  destruct(ex_conga_ts B C A C B A) as [B1 [HConga HTS]]; Col.
  assert (HPar : Par A C B B1)
     by (apply par_left_comm, par_symmetry, l12_21_b; Side; CongA).
  apply (par_not_col_strict _ _ _ _ B) in HPar; Col.
  assert(HNCol1 : ~ Col C B B1) by (apply (par_not_col A C); Col).
  assert(HNCol2 : ~ Col A B B1) by (apply (par_not_col A C); Col).
  assert(HB2 := segment_construction B1 B B1 B).
  destruct HB2 as [B2 [HBet HCong]].
  assert_diffs.

  assert(HTS1 : TS B A B1 B2).
  { repeat split; Col.
    intro; apply HNCol2; ColR.
    exists B; Col.
  }
  assert(HTS2 : TS B A C B2).
  { apply (l9_8_2 _ _ B1); auto.
    apply os_ts1324__os; Side.
  }
  apply (bet_conga_bet B1 B B2); auto.
  apply (suma2__conga D1 E1 F1 C A B); auto.
  assert(CongA A B B2 C A B).
  { apply conga_left_comm, aia; Side.
    apply par_symmetry, (par_col_par _ _ _ B1); Col; Par.
  }
  apply (conga3_suma__suma B1 B A A B B2 B1 B B2); try (apply conga_refl); auto.
    exists B2; repeat (split; CongA); apply l9_9; auto.
  apply (suma2__conga A B C B C A); auto.
  apply (conga3_suma__suma A B C C B B1 A B B1); CongA.
  exists B1; repeat (split; CongA); apply l9_9; Side.
Qed.

End alternate_interior_angles_postulate_triangle.
```

Proof of Varignon's theorem using the midpoint theorem:

```
Lemma varignon :
```

```
 forall A B C D I J K L,
  A<>C -> B<>D -> ~ Col I J K ->
  Midpoint I A B -> Midpoint J B C ->
  Midpoint K C D -> Midpoint L A D ->
  Parallelogram I J K L.
Proof.
intros.
assert_diffs.
assert (Par I L B D)
(* Applying the midpoint theorem in the triangle BDA. *)
  by perm_apply (triangle_mid_par B D A L I).
assert (Par J K B D)
(* Applying the midpoint theorem in the triangle BDC. *)
  by perm_apply (triangle_mid_par B D C K J).
assert (Par I L J K)
(* Transitivity of parallelism *)
  by (apply par_trans with B D;finish).
assert (Par I J A C)
(* Applying the midpoint theorem in the triangle ACB. *)
  by perm_apply (triangle_mid_par A C B J I).
assert (Par L K A C)
(* Applying the midpoint theorem in the triangle ACD. *)
  by perm_apply (triangle_mid_par A C D K L).
assert (Par I J K L)
(* Transitivity of parallelism *)
  by (apply par_trans with A C;finish).
apply par_2_plg;finish.
(* If in the opposite side of quadrilateral are parallel and
   two opposite side are distinct then it is a parallelogram. *)
Qed.
```

Alternative proof using characterisation of parallelogram using midpoints and coordinates:

```
Lemma varignon : forall A B C D I J K L,
 A<>B -> A<>C -> A<>D ->
 Midpoint I A B -> Midpoint J B C ->
 Midpoint K C D -> Midpoint L A D ->
 Parallelogram I J K L.
Proof.
intros A B C D I J K L HAB HAC HAD HI HJ HK HL.
destruct (midpoint_existence I K) as [O HO].
assert (I<>K \/ J<>L).
 {
 destruct (eq_dec_points I K).
 subst;right.
```

```
 intro.
 treat_equalities.
  assert (Parallelogram A C B D).
   apply mid_plg with O;auto.
 assert (Parallelogram A C D B).
   apply mid_plg with J;finish.
 apply (plg_not_comm_1 A C B D);auto.
 auto.
 }
assert (Midpoint O J L).
{
 revert HI HJ HK HL HO.
 convert_to_algebra.
 decompose_coordinates;intros;spliter.
 split;
 nsatz;prove_discr_for_powers_of_2.
}
apply mid_plg with O;assumption.
Qed.
```

A completely automatic proof using the area method:

```
Theorem varignon:
 forall A B C D I J K L,
 is_midpoint I A B ->
 is_midpoint J B C ->
 is_midpoint K C D ->
 is_midpoint L D A ->
 parallel I J K L /\ parallel J K I L.
Proof.
area_method.
Qed.
```

A detailed proof script using the area method, the tactics highlights the key idea of eliminating points one by one from the goal, but the actual computation is implicit:

```
Theorem varignon:
 forall A B C D I J K L,
 is_midpoint I A B ->
 is_midpoint J B C ->
 is_midpoint K C D ->
 is_midpoint L D A ->
 parallel I J K L /\ parallel J K I L.
Proof.
geoInit.
eliminate I.
```

```
eliminate J.
eliminate K.
eliminate L.
Runiformize_signed_areas.
field_and_conclude.
eliminate I.
eliminate J.
eliminate K.
eliminate L.
Runiformize_signed_areas.
field_and_conclude.
Qed.
```